Advanced Mathematical Models & Applications
Vol.7, No.2, 2022, pp.178-190

SOLUTION OF ONE PROBLEM FOR LINEAR LOADED PARABOLIC TYPE OF DIFFERENTIAL EQUATION WITH INTEGRAL CONDITIONS

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Abstract. The finite difference method is used to solution one problem for a linear loaded differential equation of parabolic type with integral conditions. A difference problem that approximates the original problem with the second order of accuracy is constructed, under certain conditions, the convergence of the method is proved and the rate of convergence is determined. After replacing the integral conditions with non-local boundary conditions, the finite difference method is applied to the solution of the problem. The corresponding difference problem is constructed, the convergence of the method is proved under certain conditions, and the rate of convergence is determined.

Keywords: loaded differential equation, finite difference method, integral conditions, maximum principle.
AMS Subject Classification: 35K10, 35K55.
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Received: 26 April 2022; Revised: 5 July 2022; Accepted: 23 July 2022; Published: 5 August 2022.

1 Introduction

In this paper, one problem is solved for a linear loaded differential equation of parabolic type with integral conditionsby the finite difference method. It is known that many problems of natural science, for example, problems of mathematical physics and biology, problems of long-term forecasting and regulation of groundwater, problems of heat and mass transfer at a finite rate, etc. lead to problems for loaded partial differential equations. Similar problems can be found, for example, in the works of Nakhushev (1995, 2012). Therefore, the solution of such problems for loaded differential equations is of great interest.

Different problems for loaded differential equations have been studied by many mathematicians. We can also refer the papers Dzenaliev & Ramazanov (2006), Abdullaev & Aida-Zade (2016), Khankishiyev (2017, 2020), Agarwal et al., (2020), Parasidis et al. (2018), Islomov & Alikulov (2021). Among the methods for solving such problems, various numerical methods are often used. In the books of Samarsky (2001) and Samarsky & Nikolaev (1989), effective methods for solving such problems are given, and in the work of Ashyralyev & Ahmed (2019) one specific problem of an applied nature is solved by a numerical method.

There are problems for partial differential equations with integral conditions. Such conditions arise when, for example, the area of application of a concentrated force has a finite size. In the works of Khankishiev (2017, 2020) specific mixed problems were solved by the method of finite differences for a linear loaded differential equation of parabolic and hyperbolic types with integral conditions. In these papers, algorithms for solving the constructed difference problems are given and their convergence is studied.

2 Statement of the problem

The present work is devoted to solving a problem with integral conditions for the linear loaded differential equation of parabolic type. Here we study the following problem for this equation with a variable coefficient: to find continuous function $u = u(x,t)$ in closed domain $\bar{D} = \{0 \le x \le l,\ 0 \le t \le T\}$, which satisfies to equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + bu(x,t) + \sum_{k=1}^{m} b_k u(x,\bar{t}_k) + f(x,t), \quad 0 < x < l\,, \quad 0 < t \le T\,, \tag{1}$$

integral conditions

$$\int_0^l c_1(x)u(x,t)dx = \mu_1(t), \quad \int_0^l c_2(x)u(x,t)dx = \mu_2(t), \quad 0 \le t \le T, \tag{2}$$

and initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le l. \tag{3}$$

Here $k(x) \ge k_0 > 0$, $f(x,t)$, $\mu_1(t)$, $\mu_2(t)$, $c_1(x)$, $c_2(x)$, $\varphi(x)-$are known continuous functions of there arguments. b, b_k, $k = 1,2,...,m$ are real numbers, $\bar{t}_k$, $k = 1,2,...,m$ are points of the interval $(0,T]$.

In what follows, we assume that problem (1)-(3) has a unique solution.

3 Replacement of integral conditions

Consider first of conditions in (2) and differentiate with respect to t :

$$\int_0^l c_1(x)\frac{\partial u(x,t)}{\partial t}dx = \mu_1'(t).$$

Hence, by virtue of equation (1), we have:

$$\int_0^l c_1(x)\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + bu(x,t) + \sum_{k=1}^{m} b_k u(x,\bar{t}_k) + f(x,t)\right]dx = \mu_1'(t)\,, \tag{4}$$

Applying twice the formula for integration by parts to the integral of the first term, we obtain:

$$\int_0^l c_1(x)\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)dx = c_1(x)k(x)\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}^{x=l} - c_1'(x)k(x)u(x,t)\big|_{x=0}^{x=l} +$$

$$+\int_0^l \left(c_1'(x)k(x)\right)' u(x,t)dx\,.$$

Using notation $C_1(x) = \left(c_1'(x)k(x)\right)'$, the last equality can be rewritten as follows:

$$\int_0^l c_1(x)\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)dx = c_1(x)k(x)\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}^{x=l} - c_1'(x)k(x)u(x,t)\big|_{x=0}^{x=l} + \int_0^l C_1(x)u(x,t)dx\,.$$

Finally, applying the trapezoidal formula to the last integral on the right-hand side, we obtain the equality

$$\int_0^l c_1(x)\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)dx = c_1(x)k(x)\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}^{x=l} - c_1'(x)k(x)u(x,t)\big|_{x=0}^{x=l} +$$

$$+h\cdot\left[\frac{C_1(x_0)}{2}u(x_0,t)+C_1(x_1)u(x_1,t)+\cdots+C_1(x_{N-1})u(x_{N-1},t)+\ \frac{C_1(x_N)}{2}u(x_N,t)\right]+O\left(h^2\right).$$

Taking into account this equality in (4), after elementary transformations, we get the following equality:

$$c_1(x)k(x)\ \frac{\partial u(x,t)}{\partial x}\bigg|_{x=0}^{x=l}-\ c_1'(x)k(x)u(x,t)\big|_{x=0}^{x=l}+h\cdot\left[\frac{C_1(x_0)}{2}u(x_0,t)+\ C_1(x_1)u(x_1,t)+\cdots+\right.$$

$$\left.+C_1(x_{N-1})\cdot\ u(x_{N-1},t)+\ \frac{C_1(x_N)}{2}u(x_N,t)\right]=\mu_1'(t)-b\mu_1(t)-\sum_{k=1}^{m}b_k\mu_1(\bar t_k)-$$

$$-\int_0^l c_1(x)f(x,t)+O\left(h^2\right)\ . \tag{5}$$

Similarly, we have

$$c_2(x)k(x)\ \frac{\partial u(x,t)}{\partial x}\bigg|_{x=0}^{x=l}-\ c_2'(x)k(x)u(x,t)\big|_{x=0}^{x=l}+h\cdot\left[\frac{C_2(x_0)}{2}u(x_0,t)+\ C_2(x_1)u(x_1,t)+\cdots+\right.$$

$$\left.+\,C_2(x_{N-1})u(x_{N-1},t)+\ \frac{C_2(x_N)}{2}u(x_N,t)\right]=\mu_2'(t)-b\mu_2(t)-\sum_{k=1}^{m}b_k\mu_2(\bar t_k)-$$

$$-\int_0^l c_2(x)f(x,t)+O\left(h^2\right), \tag{6}$$

here $C_2(x)=(c_2'(x)k(x))'$.

And so, instead of integral conditions (2), we obtained conditions (5) and (6).

If we exclude from the conditions (5) and (6), first $\frac{\partial u(l,t)}{\partial x}$, then $\frac{\partial u(0,t)}{\partial x}$, then instead of these conditions we get the conditions

$$\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)k(0)\frac{\partial u(0,t)}{\partial x}-\left(c_1'(0)c_2(l)-c_1(l)c_2'(0)\right)k(0)u(0,t)+$$

$$+\left(c_1'(l)c_2(l)-c_1(l)c_2'(l)\right)k(l)u(l,t)+h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{2}\ u(x_0,t)+\right.$$

$$+\left(c_1(l)C_2(x_1)-C_1(x_1)c_2(l)\right)u(x_1,t)+\cdots+\left(c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)\right)u(x_{N-1},t)+$$

$$\left.+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2}u(x_N,t)\right]=\bar\mu_1(t)+O\left(h^2\right), \tag{7}$$

$$\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)k(l)\frac{\partial u(l,t)}{\partial x}-\left(c_1'(0)c_2(0)-c_1(0)c_2'(0)\right)k(0)u(0,t)+$$

$$+\left(c_1'(l)c_2(0)-c_1(0)c_2'(l)\right)k(l)u(l,t)+h\left[\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2}\ u(x_0,t)+\right.$$

$$+\left(c_1(0)C_2(x_1)-C_1(x_1)c_2(0)\right)u(x_1,t)+\cdots+\left(c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)\right)u(x_{N-1},t)+$$

$$\left.+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2}u(x_N,t)\right]=\bar\mu_2(t)+O\left(h^2\right), \tag{8}$$

where

$$\bar\mu_1(t)=c_1(l)\mu_2'(t)-c_2(l)\mu_1'(t)+b\left(c_2(l)\mu_1(t)-c_1(l)\mu_2(t)\right)+c_2(l)\sum_{k=1}^{m}b_k\mu_1(\bar t_k)-$$

$$-c_1(l)\sum_{k=1}^{m} b_k\mu_2(\bar t_k)+c_2(l)\int_0^l c_1(x)f(x,t)dx-c_1(l)\int_0^l c_2(x)f(x,t)dx,$$

$$\bar\mu_2(t)=c_1(0)\mu_2'(t)-c_2(0)\mu_1'(t)+b\left(c_2(0)\mu_1(t)-c_1(0)\mu_2(t)\right)+c_2(0)\sum_{k=1}^{m} b_k\mu_1(\bar t_k)-$$

$$-c_1(0)\sum_{k=1}^{m} b_k\mu_2(\bar t_k)+c_2(0)\int_0^l c_1(x)f(x,t)dx-c_1(0)\int_0^l c_2(x)f(x,t)dx.$$

Suppose, that $c_1(0)c_2(l)-c_1(l)c_2(0)\neq 0$. Then, dividing both sides of equalities (7) and (8) by this expression, we obtain:

$$k(0)\frac{\partial u(0,t)}{\partial x}-\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)u(0,t)+\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)u(l,t)+$$

$$+h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\,u(x_0,t)+\frac{c_1(l)C_2(x_1)-C_1(x_1)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_1,t)+\dots+\right.$$

$$\left.+\frac{c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_{N-1},t)+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}u(x_N,t)\right]=$$

$$=\bar{\bar\mu}_1(t)+O\left(h^2\right), \tag{9}$$

$$k(l)\frac{\partial u(l,t)}{\partial x}-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)u(0,t)+\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)u(l,t)+$$

$$+h\left[\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\,u(x_0,t)+\frac{c_1(0)C_2(x_1)-C_1(x_1)c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_1,t)+\dots+\right.$$

$$\left.+\frac{c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_{N-1},t)+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}u(x_N,t)\right]=$$

$$=\bar{\bar\mu}_2(t)+O\left(h^2\right)\,, \tag{10}$$

where

$$\bar{\bar\mu}_1(t)=\frac{\bar\mu_1(t)}{c_1(0)c_2(l)-c_1(l)c_2(0)},\ \ \bar{\bar\mu}_2(t)=\frac{\bar\mu_2(t)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\,.$$

From Taylor formula we'll get:

$$k(x)\frac{\partial u(x,t)}{\partial x}=k(0)\frac{\partial u(0,t)}{\partial x}+x\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right)\right]_{x=0}+O\left(x^2\right).$$

Taking in this equality $x=\frac{h}{2}$, we can easily obtain the validity of the equality

$$k(0)\frac{\partial u(0,t)}{\partial x}=k\left(\frac{h}{2}\right)\frac{u(x_1,t)-u(0,t)}{h}-\frac{h}{2}\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right)\right]_{x=0}+O\left(h^2\right).$$

By the same way, we obtain the validity of the equality

$$k(l)\frac{\partial u(l,t)}{\partial x}=k\left(l-\frac{h}{2}\right)\frac{u(x_N,t)-u(x_{N-1},t)}{h}+\frac{h}{2}\left[\frac{\partial}{\partial x}\left(k(x)\frac{\partial u(x,t)}{\partial x}\right)\right]_{x=l}+O\left(h^2\right).$$

Assuming the fulfillment of equation (1) on the boundaries $x=0$ and $x=l$ of the domain $\bar D$, from the last two equalities we obtain:

$$k(0)\frac{\partial u(0,t)}{\partial x}=k\left(\frac{h}{2}\right)\frac{u(x_1,t)-u(0,t)}{h}-\frac{h}{2}\left[\frac{\partial u(0,t)}{\partial t}-bu(0,t)-\sum_{k=1}^{m} b_k u(0,\bar t_k)-f(0,t)\right]+O\left(h^2\right),$$

$$k(l)\frac{\partial u(l,t)}{\partial x}=k\left(l-\frac{h}{2}\right)\frac{u(x_N,t)-u(x_{N-1},t)}{h}+$$
$$+\frac{h}{2}\left[\frac{\partial u(l,t)}{\partial t}-bu(l,t)-\sum_{k=1}^{m}b_k u(l,\bar t_k)-f(l,t)\right]+O\left(h^2\right).$$

Taking into account these equalities in (9) and (10), correspondingly, we arrive at the equalities

$$k\left(\frac{h}{2}\right)\frac{u(x_1,t)-u(0,t)}{h}-\frac{h}{2}\left[\frac{\partial u(0,t)}{\partial t}-bu(0,t)-\sum_{k=1}^{m}b_k u(0,\bar t_k)-f(0,t)\right]-$$
$$-\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)u(0,t)+\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)u(l,t)+$$
$$+h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}u(x_0,t)+\frac{c_1(l)C_2(x_1)-C_1(x_1)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_1,t)+\dots+\right.$$
$$\left.+\frac{c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_{N-1},t)+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}u(x_N,t)\right]=$$
$$=\tilde\mu_1(t)+O\left(h^2\right), \tag{11}$$

$$k\left(l-\frac{h}{2}\right)\frac{u(x_N,t)-u(x_{N-1},t)}{h}+\frac{h}{2}\left[\frac{\partial u(l,t)}{\partial t}-bu(l,t)-\sum_{k=1}^{m}b_k u(l,\bar t_k)-f(l,t)\right]-$$
$$-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)u(0,t)+\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)u(l,t)+$$
$$+h\left[\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}u(x_0,t)+\frac{c_1(0)C_2(x_1)-C_1(x_1)c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_1,t)+\dots+\right.$$
$$\left.+\frac{c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}u(x_{N-1},t)+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}u(x_N,t)\right]=$$
$$=\tilde\mu_2(t)+O\left(h^2\right), \tag{12}$$

where

$$\tilde\mu_1(t)=\bar{\bar\mu}_1(t)-\frac{h}{2}f(0,t),\quad \tilde\mu_2(t)=\bar{\bar\mu}_2(t)+\frac{h}{2}f(l,t).$$

4 Difference problem

Divide the segment [0, l] of the axis Ox by points $x_n=nh$, $n=0,1,2,...,N$, $h=l/N$, into N equal parts, and the segment [0,T] of the axis Ot by points $t_j=j\tau$, $j=0,1,2,...,j_0$, $\tau=T/j_0$, into j_0 equal parts. We choose the step τ in such a way that the points $\bar t_k$, $k=1,2,...,m$, are among the points $t_j=j\tau$, $j=1,2,...,j_0$. Suppose, that $\bar t_k=t_{j_k}$, $k=1,2,...,m$, $t_{j_1}<t_{j_2}<...<t_{j_m}$. Define in the domain $\bar D$ a grid $\bar\omega_{h\tau}=\{(x_n\,,\,t_j),\ n=0,1,2,...,N, j=0,1,2,...,j_0\}$.

Using instead of integral conditions (2) the last non-local conditions (11) and (12), to problem (1) - (3) on the grid $\bar\omega_{h\tau}$ we can associate the following difference problem:

$$\frac{h}{2}\frac{y_0^{j+1}-y_0^j}{\tau}-\frac{1}{2}k\left(\frac{h}{2}\right)\left(\frac{y_1^{j+1}-y_0^{j+1}}{h}+\frac{y_1^j-y_0^j}{h}\right)-\frac{h}{2}\left(b\frac{y_0^{j+1}+y_0^j}{2}+\sum_{k=1}^{m}b_k y_0^{j_k}\right)+$$
$$+\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)\frac{y_0^{j+1}+y_0^j}{2}-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\frac{y_N^{j+1}+y_N^j}{2}-$$

$$-h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_0^{j+1}+y_0^j}{2}+\frac{c_1(l)C_2(x_1)-C_1(x_1)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_1^{j+1}+y_1^j}{2}+\cdots+\right.$$

$$\left.+\frac{c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_{N-1}^{j+1}+y_{N-1}^j}{2}+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_N^{j+1}+y_N^j}{2}\right]=\ f_0^j,$$

$$\frac{y_n^{j+1}-y_n^j}{\tau}-\frac{1}{2}k\left(x_n+\frac{h}{2}\right)\frac{y_{n+1}^{j+1}-y_n^{j+1}+y_{n+1}^j-y_n^j}{h^2}+\frac{1}{2}k\left(x_n-\frac{h}{2}\right)\frac{y_n^{j+1}-y_{n-1}^{j+1}+y_n^j-y_{n-1}^j}{h^2}-$$

$$-b\frac{y_n^{j+1}+y_n^j}{2}-\sum_{k=1}^{m}b_k y_n^{j_k}=f_n^j,\ \ n=1,2,...,N-1, \tag{13}$$

$$\frac{h}{2}\frac{y_N^{j+1}-y_N^j}{\tau}+\frac{1}{2}k\left(l-\frac{h}{2}\right)\left(\frac{y_N^{j+1}-y_{N-1}^{j+1}}{h}+\frac{y_N^j-y_{N-1}^j}{h}\right)-\frac{h}{2}\left(b\frac{y_N^{j+1}+y_N^j}{2}+\sum_{k=1}^{m}b_k y_N^{j_k}\right)-$$

$$-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)\frac{y_0^{j+1}+y_0^j}{2}+\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\frac{y_N^{j+1}+y_N^j}{2}+$$

$$+h\left[\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_0^{j+1}+y_0^j}{2}+\frac{c_1(0)C_2(x_1)-C_1(x_1)c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_1^{j+1}+y_1^j}{2}+\cdots+\right.$$

$$\left.+\frac{c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_{N-1}^{j+1}+y_{N-1}^j}{2}+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_N^{j+1}+y_N^j}{2}\right]=$$

$$=f_N^j,\ j=0,1,...,j_0-1,$$

$$y_n^0\ =\varphi(x_n),\ \ n=0,1,...,N, \tag{14}$$

where

$$f_0^j=-\tilde{\mu}_1\left(t_j+0,5\tau\right)+\frac{h}{2}f\left(0,\ t_j+0,5\tau\right),\ f_N^j=\tilde{\mu}_2\left(t_j+0,5\tau\right)+\frac{h}{2}f\left(l,\ t_j+0,5\tau\right),$$

$$f_n^j=f\left(x_n,\ t_j+0,5\tau\right),\ \ n=1,2,...,N-1.$$

It should be noted that the difference problem (13)-(14) approximates problem (1)-(3) with an accuracy $O\left(h^2+\tau^2\right)$, if the solution $u(x,t)$ of problem (1) - (3) has bounded partial derivatives into domain $\bar{D}$ with respect to the variable x up to the fourth order , and with respect to the variable t up to the third order and equation is fulfilled both on the boundaries $x=0$ and $x=l$ of domain $\bar{D}$.

The solution of difference problems of the form (13)-(14) is described, for example, in (2020 b) by the author. Therefore, we will not dwell on solving this difference problem.

5 Maximum principle and consequences

Consider the difference problem (13)-(14) and prove the validity of the following theorem (maximum principle) with respect to the solution of this problem.

Theorem 1 (Maximum principle)**.** *Let the grid function*y_n^j, $n=0,1,...,N, j=0,1,...,j_0$, *satisfies problem* (13)-(14). *Let the conditions* $f_n^j\le 0$ $(f_n^j\ge 0)$, $n=0,1,...,N, j=0,1,...,j_0-1$ *are satisfies. If the following conditions are fulfilled*

$$0<k_0\le k(x)\le k_1,\ b_k>0,\ k=1,2,...,m,\ b+\sum_{k=1}^{m}b_k\le 0,$$

$$\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)} \geq 0, \quad \frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)} \geq 0,$$

$$\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0) - \frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l) \geq \varepsilon > 0,$$

$$\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l) - \frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0) \geq \delta > 0, \tag{15}$$

$$\tau \leq \min\left\{ \frac{2h^2}{2k_1 - bh^2}\ ,\ h^2 \cdot \left[k_0 + \frac{bh^2}{2} + hk(0)\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right]^{-1}, \right.$$

$$\left. h^2 \cdot \left[k_0 - \frac{bh^2}{2} + hk(l)\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right]^{-1} \right\}.$$

Then solution y_n^j , $n = 0, 1, ..., N,\ j = 0, 1, ..., j_0$, *of the problem* (13)-(14)*, differ from constant, cannot take the largest positive (smallest negative) value at* $n = 0, 1, ..., N,\ j = 1, 2, ..., j_0$.

Proof. Let prove the first part of the theorem. Suppose $f_n^j \leq 0$, $n = 0, 1, ..., N,\ j = 0, 1, ..., j_0 - 1$, and conditions (15) are satisfied, but the solution y_n^j of the problem (13)-(14) takes the largest positive value at $n = n_0,\ j = i + 1,\ 0 \leq n_0 \leq N,\ 0 \leq i \leq j_0 - 1$:

$$y_{n_0}^{i+1} = \max_{0 \leq n \leq N,\, 0 \leq j \leq j_0} y_n^j = M > 0.$$

Suppose, that $0 < n_0 < N$. Without loss of generality, we can assume that $y_{n_0}^{i+1} > y_{n_0-1}^{i+1}$. Consider the difference equation in (13) at $n = n_0,\ j = i$:

$$f_{n_0}^i = -\frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right)y_{n_0-1}^{i+1} + \left(\frac{1}{\tau} + \frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right) + \frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right) - \frac{b}{2}\right)y_{n_0}^{i+1} -$$

$$-\frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right)y_{n_0+1}^{i+1} - \frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right)y_{n_0-1}^{i} +$$

$$+\left(-\frac{1}{\tau} + \frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right) + \frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right) - \frac{b}{2}\right)\cdot$$

$$\cdot y_{n_0}^i - \frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right)y_{n_0+1}^i - \sum_{k=1}^m b_k y_{n_0}^{j_k} > -\frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right)M + \left(\frac{1}{\tau} + \frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right) + \right.$$

$$\left. + \frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right) - \frac{b}{2}\right)M -$$

$$-\frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right)M - \frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right)M + \left(-\frac{1}{\tau} + \frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right) + \right.$$

$$\left. + \frac{1}{2h^2}k\left(x_{n_0} - \frac{h}{2}\right) - \frac{b}{2}\right)M - \frac{1}{2h^2}k\left(x_{n_0} + \frac{h}{2}\right)M - M\sum_{k=1}^m b_k = -\left(b + \sum_{k=1}^m b_k\right)M \geq 0,$$

since by the hypothesis of the theorem $b + \sum_{k=1}^m b_k \leq 0$. This contradicts the condition $f_{n_0}^i \leq 0$.

Suppose, that $n_0 = 0$. Without loss of generality, we can assume that $y_0^{i+1} > y_1^{i+1}$. Consider the first equation in (13) at $j = i$:

$$f_0^i = \left(\frac{h}{2\tau} + \frac{1}{2h}k\left(\frac{h}{2}\right) - \frac{bh}{4} + \frac{k(0)}{2}\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right)y_0^{i+1} - \left(\frac{h}{2\tau} - \frac{1}{2h}k\left(\frac{h}{2}\right) + \frac{bh}{4} - \right.$$

$$-\frac{k(0)}{2}\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\Bigg) y_0^i-\frac{1}{2h}k\left(\frac{h}{2}\right)y_1^{i+1}-\frac{1}{2h}k\left(\frac{h}{2}\right)y_1^i-\frac{h}{2}\sum_{k=1}^{m} b_k y_0^{j_k}-$$

$$-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\frac{y_N^{i+1}+y_N^i}{2}-h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_0^{i+1}+y_0^i}{2}+\right.$$

$$+\frac{c_1(l)C_2(x_1)-C_1(x_1)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_1^{i+1}+y_1^i}{2}+\cdots+\frac{c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_{N-1}^{i+1}+y_{N-1}^i}{2}+$$

$$\left.+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_N^{i+1}+y_N^i}{2}\right].$$

Hence, under conditions (15), after simple transformations, we obtain:

$$f_0^i>\varepsilon M-h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_0^{i+1}+y_0^i}{2}+\frac{c_1(l)C_2(x_1)-C_1(x_1)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_1^{i+1}+y_1^i}{2}+\right.$$

$$\left.+...+\frac{c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_{N-1}^{i+1}+y_{N-1}^i}{2}+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_N^{i+1}+y_N^i}{2}\right].$$

To evaluate an expression in square brackets, suppose that

$$\min_{0\le n\le N,\,0\le j\le j_0} y_n^j=M_1.$$

In this case, replace $\frac{y_k^{i+1}+y_k^i}{2}$ through M, if $\frac{c_1(l)C_2(x_k)-C_1(x_k)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\ge 0$, $k=0,1,...,N$, and through M_1 otherwise. After that, denoting the sum of all the terms in these brackets through αM,from the previous inequality, we obtain: $f_0^i>\varepsilon M-h\alpha M=(\varepsilon-\alpha h)M>0$, if $\alpha\le 0$, but if $\alpha>0$, then this inequality takes place at $h\le\frac{\varepsilon}{\alpha}$. This contradicts the condition $f_0^i\le 0$.

Suppose, that $n_0=N$. Without loss of generality, we can assume that $y_N^{i+1}>y_{N-1}^{i+1}$. Consider the last equation in (13) at $j=i$:

$$f_N^i=\left(\frac{h}{2\tau}+\frac{1}{2h}k\left(l-\frac{h}{2}\right)-\frac{bh}{4}+\frac{k(l)}{2}\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right)y_N^{i+1}-\left(\frac{h}{2\tau}-\frac{1}{2h}k\left(l-\frac{h}{2}\right)+\right.$$

$$\left.+\frac{bh}{4}-\frac{k(l)}{2}\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\right)y_N^i-\frac{1}{2h}k\left(l-\frac{h}{2}\right)y_{N-1}^{i+1}-\frac{1}{2h}k\left(l-\frac{h}{2}\right)y_{N-1}^i-\frac{h}{2}\sum_{k=1}^{m} b_k y_N^{j_k}-$$

$$-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)\frac{y_0^{i+1}+y_0^i}{2}+h\left[\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_0^{i+1}+y_0^i}{2}+\right.$$

$$+\frac{c_1(0)C_2(x_1)-C_1(x_1)c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_1^{i+1}+y_1^i}{2}+\cdots+\frac{c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_{N-1}^{i+1}+y_{N-1}^i}{2}+$$

$$\left.+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_N^{i+1}+y_N^i}{2}\right].$$

From this, as in the case $n_0=0$, under conditions (15) we obtain:

$$f_N^i>\delta M+h\left[\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_0^{i+1}+y_0^i}{2}+\frac{c_1(0)C_2(x_1)-C_1(x_1)c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_1^{i+1}+y_1^i}{2}+\right.$$

$$\left.+...+\frac{c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{y_{N-1}^{i+1}+y_{N-1}^i}{2}+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{y_N^{i+1}+y_N^i}{2}\right]\ge$$

$\ge\delta M+h\beta M=(\delta+h\beta)M>0$, at $\beta\ge 0$. Otherwise, at $h>-\frac{\delta}{\beta}$.

The first part of the theorem is proved. The second part of the theorem can be proved in a similar way. ☐

Theorem 2. *Suppose, that grid function* y_n^j, $n = 0,1,...,N,$ $j = 0,1,...j_0$, *satisfies to problem* (13) - (14). *If* $f_n^j \leq 0,\quad \varphi(x_n) \leq 0 \quad \left(f_n^j \geq 0,\ \varphi(x_n) \geq 0\right)$,$n = 0,1,...,N$,$j = 0,1,...,j_0 - 1$ *and conditions* (15) *are satisfies, then* $y_n^j \leq 0 \ \left(y_n^j \geq 0\right)$, $n = 0,1,...,N,$ $j = 0,1,...j_0$.

The correctness of the statement of this theorem follows from the maximum principle.

Consequence. Let conditions (15) are satisfied. Then the homogeneous problem corresponding to problem (13)-(14) has only the trivial solution $y_n^j = 0,$ $n = 0,1,...,N,$ $j = 0,1,...,j_0$.

It follows from this corollary that under conditions (15) there exists a unique solution to the difference problem (13)–(14).

Theorem 3 ((Comparison theorem))**.** *Suppose, that* y_n^j, $n = 0,1,...,N,$ $j = 0,1,...j_0-$ *solution of the difference problem* (13)-(14) *and* $\tilde{y}_n^j$, $n = 0,1,...,N, j = 0,1,...j_0-$*solution of the difference problem obtained by replacing in* (13)-(14) *the functions*

f_n^j, $n = 0,1,...,N,$ $j = 0,1,2,...,j_0 - 1,$ *and* $\varphi(x_n), n = 0,1,...,N,$ *correspondingly, by* $\tilde{f}_n^j$, $n = 0,1,...,N,$ $j = 0,1,2,...,j_0 - 1,$ *and* $\tilde{\varphi}(x_n), n = 0,1,...,N.$ *Then, if* $\left|f_n^j\right| \leq \tilde{f}_n^j$, $n = 0,1,...,N,$ $j = 0,1,...,j_0 - 1,$ *and* $|\varphi(x_n)| \leq \tilde{\varphi}(x_n)$, $n = 0,1,...,N,$ *then under conditions (15) inequalities* $\left|y_n^j\right| \leq \tilde{y}_n^j$, $n = 0,1...,N,$ $j = 0,1,...,j_0$ *take place.*

Remark 1. *The set of functions* $k(x),$ $c_1(x),$ $c_2(x)$,*satisfying conditions* (15) *is not an empty set. Really, suppose, that*

$$l = 1,\ c_1(x) = \frac{1}{2}\left(-x^3 + x^2 + x + 1\right),\quad c_2(x) = -7x^3 + 11x^2 - 4x - \frac{1}{2}.$$

Then

$$\frac{c_1'(1)c_2(1) - c_1(1)c_2'(1)}{c_1(0)c_2(1) - c_1(1)c_2(0)} = 12,\quad \frac{c_1'(0)c_2(0) - c_1(0)c_2'(0)}{c_1(0)c_2(1) - c_1(1)c_2(0)} = 7,$$

$$\frac{c_1'(0)c_2(1) - c_1(1)c_2'(0)}{c_1(0)c_2(1) - c_1(1)c_2(0)} = 15,\quad \frac{c_1'(1)c_2(0) - c_1(0)c_2'(1)}{c_1(0)c_2(1) - c_1(1)c_2(0)} = 6,$$

$$\frac{c_1'(0)c_2(1) - c_1(1)c_2'(0)}{c_1(0)c_2(1) - c_1(1)c_2(0)}k(0) - \frac{c_1'(1)c_2(1) - c_1(1)c_2'(1)}{c_1(0)c_2(1) - c_1(1)c_2(0)}k(1) = 15k(0) - 12k(1),$$

$$\frac{c_1'(1)c_2(0) - c_1(0)c_2'(1)}{c_1(0)c_2(1) - c_1(1)c_2(0)}k(1) - \frac{c_1'(0)c_2(0) - c_1(0)c_2'(0)}{c_1(0)c_2(1) - c_1(1)c_2(0)}k(0) = 6k(1) - 7k(0).$$

We require the fulfillment of the inequalities

$$15k(0) - 12k(1) \geq \varepsilon, 6k(1) - 7k(0) \geq \delta,$$

where $\varepsilon > 0,$ $\delta > 0-$*arbitrary numbers. From the last two inequalities, we get that*

$$k(0) \geq \varepsilon + 2\delta,\quad k(1) \geq \frac{7\varepsilon + 15\delta}{6}.$$

This means that the set of functions $k(x),$ $c_1(x),$ $c_2(x)$, satisfying the conditions (15) is, indeed, not an empty set.

6 Convergence

In the grid domain $\bar{\omega}_{h\tau}$ we define the grid function z_n^j by the equality

$$z_n^j = y_n^j - u(x_n, t_j), \;\; n = 0, 1, ..., N, \;\; j = 0, 1, ..., j_0,$$

where y_n^j solution of the difference problem (13)-(14), $u(x_n,\, t_j)$- the value of the exact solution to problem (1) - (3) at the grid point$(x_n,\, t_j)$ of grid $\bar{\omega}_{h\tau}$. If we substitute the expression y_n^j found from the last equality in the difference problem (13) - (14), then with respect to the function z_n^j we obtain:

$$\frac{h}{2}\frac{z_0^{j+1} - z_0^j}{\tau} - \frac{1}{2}k\left(\frac{h}{2}\right)\left(\frac{z_1^{j+1} - z_0^{j+1}}{h} + \frac{z_1^j - z_0^j}{h}\right) - \frac{h}{2}\left(b\frac{z_0^{j+1} + z_0^j}{2} + \sum_{k=1}^{m} b_k z_0^{j_k}\right) +$$

$$+\frac{c_1'(0)c_2(l) - c_1(l)c_2'(0)}{c_1(0)c_2(l) - c_1(l)c_2(0)}k(0)\frac{z_0^{j+1} + z_0^j}{2} - \frac{c_1'(l)c_2(l) - c_1(l)c_2'(l)}{c_1(0)c_2(l) - c_1(l)c_2(0)}k(l)\frac{z_N^{j+1} + z_N^j}{2} -$$

$$-h\left[\frac{c_1(l)C_2(x_0) - C_1(x_0)c_2(l)}{2\left(c_1(0)c_2(l) - c_1(l)c_2(0)\right)}\,\frac{z_0^{j+1} + z_0^j}{2} + \frac{c_1(l)C_2(x_1) - C_1(x_1)c_2(l)}{c_1(0)c_2(l) - c_1(l)c_2(0)}\frac{z_1^{j+1} + z_1^j}{2} + \cdots + \right.$$

$$\left. + \frac{c_1(l)C_2(x_{N-1}) - C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l) - c_1(l)c_2(0)}\frac{z_{N-1}^{j+1} + z_{N-1}^j}{2} + \frac{c_1(l)C_2(x_N) - C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l) - c_1(l)c_2(0)\right)}\frac{z_N^{j+1} + z_N^j}{2}\right] = \psi_0^j,$$

$$\frac{z_n^{j+1} - z_n^j}{\tau} - \frac{1}{2}k\left(x_n + \frac{h}{2}\right)\frac{z_{n+1}^{j+1} - z_n^{j+1} + z_{n+1}^j - z_n^j}{h^2} + \frac{1}{2}k\left(x_n - \frac{h}{2}\right)\frac{z_n^{j+1} - z_{n-1}^{j+1} + z_n^j - z_{n-1}^j}{h^2} -$$

$$-b\frac{z_n^{j+1} + z_n^j}{2} - \sum_{k=1}^{m} b_k z_n^{j_k} = \psi_n^j, \;\; n = 1, 2, ..., N-1, \tag{16}$$

$$\frac{h}{2}\frac{z_N^{j+1} - z_N^j}{\tau} + \frac{1}{2}k\left(l - \frac{h}{2}\right)\left(\frac{z_N^{j+1} - z_{N-1}^{j+1}}{h} + \frac{z_N^j - z_{N-1}^j}{h}\right) - \frac{h}{2}\left(b\frac{z_N^{j+1} + z_N^j}{2} + \sum_{k=1}^{m} b_k z_N^{j_k}\right) -$$

$$-\frac{c_1'(0)c_2(0) - c_1(0)c_2'(0)}{c_1(0)c_2(l) - c_1(l)c_2(0)}k(0)\frac{z_0^{j+1} + z_0^j}{2} + \frac{c_1'(l)c_2(0) - c_1(0)c_2'(l)}{c_1(0)c_2(l) - c_1(l)c_2(0)}k(l)\frac{z_N^{j+1} + z_N^j}{2} +$$

$$+h\left[\frac{c_1(0)C_2(x_0) - C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l) - c_1(l)c_2(0)\right)}\,\frac{z_0^{j+1} + z_0^j}{2} + \frac{c_1(0)C_2(x_1) - C_1(x_1)c_2(0)}{c_1(0)c_2(l) - c_1(l)c_2(0)}\frac{z_1^{j+1} + z_1^j}{2} + \cdots + \right.$$

$$\left. + \frac{c_1(0)C_2(x_{N-1}) - C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l) - c_1(l)c_2(0)}\frac{z_{N-1}^{j+1} + z_{N-1}^j}{2} + \frac{c_1(0)C_2(x_N) - C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l) - c_1(l)c_2(0)\right)}\frac{z_N^{j+1} + z_N^j}{2}\right] =$$

$$= \psi_N^j, \;\; j = 0, 1, ..., j_0 - 1,$$

$$z_n^0 = 0, \;\; n = 0, 1, ..., N. \tag{17}$$

here ψ_n^j, $n = 0, 1, ..., N-$ determine the error of approximation of the difference problem (13)-(14). The right-hand sides of difference equations (16) satisfy the estimate

$$\left|\psi_n^j\right| \le L\left(h^2 + \tau^2\right), \;\; n = 0, 1, ..., N, \;\; j = 0, 1, ..., j_0 - 1,$$

where $L > 0$is a constant.

Let's define the grid function

$$\tilde{z}_n^j = L\xi\left(h^2 + \tau^2\right)\,(2l_1 - x_n), \;\; n = 0, 1, ..., N, \;\; j = 0, 1, ..., j_0, \tag{18}$$

on the grid $\bar{\omega}_{h\tau}$, where $\xi > 0\,,\ l_1 \geq l$ are constants. Obviously, $\tilde{z}_n^j$ is a positive function. For this function, under conditions (15), after simple transformations we have:

$$\frac{h}{2}\frac{\tilde{z}_0^{j+1}-\tilde{z}_0^j}{\tau}-\frac{1}{2}k\left(\frac{h}{2}\right)\left(\frac{\tilde{z}_1^{j+1}-\tilde{z}_0^{j+1}}{h}+\frac{\tilde{z}_1^{j}-\tilde{z}_0^{j}}{h}\right)-\frac{h}{2}\left(b\frac{\tilde{z}_0^{j+1}+\tilde{z}_0^j}{2}+\sum_{k=1}^m b_k\tilde{z}_0^{j_k}\right)+$$

$$+\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)\frac{\tilde{z}_0^{j+1}+\tilde{z}_0^j}{2}-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\frac{\tilde{z}_N^{j+1}+\tilde{z}_N^j}{2}-$$

$$-h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\,\frac{\tilde{z}_0^{j+1}+\tilde{z}_0^j}{2}+\frac{c_1(l)C_2(x_1)-C_1(x_1)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\tilde{z}_1^{j+1}+\tilde{z}_1^j}{2}+\cdots+\right.$$

$$\left.+\frac{c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\tilde{z}_{N-1}^{j+1}+\tilde{z}_{N-1}^j}{2}+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{\tilde{z}_N^{j+1}+\tilde{z}_N^j}{2}\right]=$$

$$=L\xi\left(h^2+\tau^2\right)\left[k\left(\frac{h}{2}\right)-hl_1\left(b+\sum_{k=1}^m b_k\right)+\left(\frac{c_1'(0)c_2(l)-c_1(l)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)-\right.\right.$$

$$\left.\left.-\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\right)\cdot 2\,l_1+\frac{c_1'(l)c_2(l)-c_1(l)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\cdot l\right]-$$

$$-h\left[\frac{c_1(l)C_2(x_0)-C_1(x_0)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\,l_1+\frac{c_1(l)C_2(x_1)-C_1(x_1)c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(2l_1-h)+\cdots+\right.$$

$$\left.+\frac{c_1(l)C_2(x_{N-1})-C_1(x_{N-1})c_2(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\,(2l_1-l+h)+\frac{c_1(l)C_2(x_N)-C_1(x_N)c_2(l)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}(2l_1-l)\right]\geq$$

$$\geq L\xi\left(h^2+\tau^2\right)\left[k\left(\frac{h}{2}\right)+2l_1\varepsilon-h\alpha_1\right]\geq L\left(h^2+\tau^2\right)\,, \tag{19}$$

at $\xi\geq\left[k\left(\frac{h}{2}\right)+2l_1\varepsilon-h\alpha_1\right]^{-1}$. Here α_1 is the value of the expression in square brackets.

Let $0<n<N$. Suppose, that $b+\sum\limits_{k=1}^m b_k\leq -B<0$. Then for this function we get:

$$\frac{\tilde{z}_n^{j+1}-\tilde{z}_n^j}{\tau}-\frac{1}{2}k\left(x_n+\frac{h}{2}\right)\frac{\tilde{z}_{n+1}^{j+1}-\tilde{z}_n^{j+1}+\tilde{z}_{n+1}^j-\tilde{z}_n^j}{h^2}+\frac{1}{2}k\left(x_n-\frac{h}{2}\right)\frac{\tilde{z}_n^{j+1}-\tilde{z}_{n-1}^{j+1}+\tilde{z}_n^j-\tilde{z}_{n-1}^j}{h^2}-$$

$$-b\frac{\tilde{z}_n^{j+1}+\tilde{z}_n^j}{2}-\sum_{k=1}^m b_k\tilde{z}_n^{j_k}=L\xi\left(h^2+\tau^2\right)\left[k'(x_n)-(2l_1-x_n)\left(b+\sum_{k=1}^m b_k\right)\right]\geq \tag{20}$$

$$\geq\ L\xi\left(h^2+\tau^2\right)\left[k'(x_n)+(2l_1-l)B\right]\geq L\left(h^2+\tau^2\right)\,,$$

if $\inf\limits_{0<x<l}\left[k'(x)+(2l_1-l)B\right]>0$ and $\xi\geq\frac{1}{\inf\limits_{0<x<l}\left[k'(x)+(2l_1-l)B\right]}$.

$$\frac{h}{2}\frac{\tilde{z}_N^{j+1}-\tilde{z}_N^j}{\tau}+\frac{1}{2}k\left(l-\frac{h}{2}\right)\left(\frac{\tilde{z}_N^{j+1}-\tilde{z}_{N-1}^{j+1}}{h}+\frac{\tilde{z}_N^{j}-\tilde{z}_{N-1}^{j}}{h}\right)-\frac{h}{2}\left(b\frac{\tilde{z}_N^{j+1}+\tilde{z}_N^j}{2}+\sum_{k=1}^m b_k\tilde{z}_N^{j_k}\right)-$$

$$-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)\frac{\tilde{z}_0^{j+1}+\tilde{z}_0^j}{2}+\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\frac{\tilde{z}_N^{j+1}+\tilde{z}_N^j}{2}+$$

$$+h\left[\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\,\frac{\tilde{z}_0^{j+1}+\tilde{z}_0^j}{2}+\frac{c_1(0)C_2(x_1)-C_1(x_1)c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\tilde{z}_1^{j+1}+\tilde{z}_1^j}{2}+\cdots+\right.$$

$$\left.+\frac{c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}\frac{\tilde{z}_{N-1}^{j+1}+\tilde{z}_{N-1}^j}{2}+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}\frac{\tilde{z}_N^{j+1}+\tilde{z}_N^j}{2}\right]=$$

$$= L\xi\left(h^2+\tau^2\right)\left[-k\left(l-\frac{h}{2}\right)-\frac{h}{2}(2l_1-l)\left(b+\sum_{k=1}^{m}b_k\right)+2l_1\left(\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)-\right.\right.$$

$$\left.-\frac{c_1'(0)c_2(0)-c_1(0)c_2'(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(0)\right)-\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\cdot l+$$

$$+h\left(\frac{c_1(0)C_2(x_0)-C_1(x_0)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}2l_1+\frac{c_1(0)C_2(x_1)-C_1(x_1)c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(2l_1-h)+\cdots+\right.$$

$$\left.\left.+\frac{c_1(0)C_2(x_{N-1})-C_1(x_{N-1})c_2(0)}{c_1(0)c_2(l)-c_1(l)c_2(0)}(2l_1-l+h)+\frac{c_1(0)C_2(x_N)-C_1(x_N)c_2(0)}{2\left(c_1(0)c_2(l)-c_1(l)c_2(0)\right)}(2l_1-l)\right)\right]\geq$$

$$\geq L\xi\left(h^2+\tau^2\right)\left[-k\left(l-\frac{h}{2}\right)+2l_1\,\delta-\frac{c_1'(l)c_2(0)-c_1(0)c_2'(l)}{c_1(0)c_2(l)-c_1(l)c_2(0)}k(l)\cdot l+h\cdot\beta_1\right]\geq,$$

$$\geq L\left(h^2+\tau^2\right), \tag{21}$$

since, by the condition $\delta>0$ and by choosing l_1, it is possible to ensure that the value of the expression in square brackets is greater than zero. Therefore, if we denote this value byγ, then if $\xi\geq\frac{1}{\gamma}$, we obtain the validity of inequality (21).

On the other hand, we have the equality

$$\tilde{z}_n^0 = L\xi\left(h^2+\tau^2\right)(2l_1-x_n),\ n=0,1,...,N,\ j=0,1,...,j_0\,. \tag{22}$$

Comparing problem (16)-(17) with problem (19)-(22), by virtue of the comparison theorem we have:

$$\left|z_n^j\right|\leq\tilde{z}_n^j,\quad n=0,1,...,N\,,\ j=0,1,2,...,j_0,$$

or

$$\left|y_n^j-u(x_n,t_j)\right|\leq L\xi\left(h^2+\tau^2\right)\cdot 2l_1,\ n=0,1,...,N,\ j=0,1,...,j_0. \tag{23}$$

So we have the following

Theorem 4. *Let the solution of equation* (1) *in domain* $D=\{0<x<l,\ 0<t\leq T\}$ *have bounded partial derivatives with respect to variable x up to the fourth order and with respect to t up to the third order, and this equation is fulfilled both on the boundaries $x=0$ and $x=l$ on the domain $\bar{D}$. If conditions* (15) *and* $b+\sum\limits_{k=1}^{m}b_k\leq -B<0$ *are satisfied, then the solution of the difference problem* (13)-(14) *converges to the solution of the problem* (1)-(3)*. Moreover, estimate* (23) *holds.*

7 Conclusion

In this paper, the method of finite differences is applied to the solution of a problem for a linear loaded differential equation of parabolic type with integral conditions. Using the trapezoidal method, the integrals included in the integral conditions are replaced by integral sums, and as a result of such a replacement, instead of integral conditions, non-local boundary conditions are obtained for the equation under consideration. A difference problem that approximates the problem with new boundary conditions with the second order of accuracy is constructed. Further, under certain conditions, the maximum principle and some other theorems are proved for the solution of the difference problem. Based on these theorems, we prove the convergence of the solution of the constructed difference problem is proved and an estimate for the rate of convergence is obtained. The method applied to the solution of the problem under consideration can be used to solve more general problems for equations of parabolic and hyperbolic types.

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